Rational functions and real Schubert calculus

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1. Introduction

We single out some problems of Schubert calculus of subspaces of codimension 2 that have the property that all their solutions are real whenever the data are real. Our arguments explore the connection between subspaces of codimension 2 and rational functions of one variable.

Let $\{W_j\}_{j=1}^q$ be a finite collection of projective subspaces in general position in the complex projective space $\mathbf{P}^d$, $a_j = \dim W_j$, and

$$1 \leq a_j \leq d - 1, \quad \sum_{j=1}^q a_j = 2d - 2. \tag{1}$$

We consider the following

Problem 1. *Enumerate subspaces $X \subset \mathbf{P}^d$ of codimension 2 which intersect each W_j non-transversally, that is*

$$\dim X \cap W_j \geq a_j - 1 \quad \text{for every} \quad j \in [1, q]. \tag{2}$$

If the given subspaces W_j are in general position then the number of such complex subspaces X is finite and can be obtained by using Pieri's formula from Schubert calculus [3, Ch. I, sect. 5].

This number turns out to be the *Kostka number* corresponding to the shape $2 \times (d - 1)$ whose definition we recall. Let $\boldsymbol{a} = (a_1, \ldots, a_q)$. Consider

the Young diagrams of the shape $2 \times (d-1)$. They consist of two rows of length $d-1$. A *semi-standard Young tableau* SSYT of shape $2 \times (d-1)$ is a filling of such a diagram by positive integers, such that an integer k appears a_k times, the entries are strictly increasing in the columns and non-decreasing in the rows. The corresponding Kostka number $K_{\boldsymbol{a}}$ is the number of such SSYT. Schubert calculus interpretation of these numbers shows that $K_{\boldsymbol{a}}$ does not change if we permute the coordinates of $\boldsymbol{a}$. For a purely combinatorial proof of this see [5, Cor. 1.2.9] or [10, Theorem 7.10.2].

In this paper we treat Problem 1 over the field of real numbers. A subspace $X \subset \mathbf{P}^d$ is called *real* if it can be defined by equations with real coefficients, or, equivalently, if X is generated by a set of real points. It follows from a general result of Sottile [7] that *there exist* configurations of real subspaces W_j such that all solutions X of Problem 1 are real. The question is to find an explicit condition on W_j which would imply that all solutions are real.

The following condition was proposed by B. and M. Shapiro (see [8, 9]). Let $E : \mathbf{P}^1 \to \mathbf{P}^d$ denote the rational normal curve; in homogeneous coordinates $E(z) = (1 : z : \ldots : z^d)$. The B. and M. Shapiro conjecture for the case of subspaces of codimension 2 says that if all W_j are osculating E at distinct real points then all solutions X of Problem 1 are real. This conjecture was proved in [1]. B. and M. Shapiro made a similar conjecture for enumerative problems involving subspaces of arbitrary codimension but in this paper we only consider the case of codimension 2.

In this paper we consider an extension of the result in [1] to the case that W_j are *spanned* by some finite sets of real points on E.

Let $\{A_j\}_{j=1}^q$ be a collection of finite sets on the circle $\mathbf{RP}^1 \subset \mathbf{P}^1$. We say that this collection is *separated* if there exist pairwise disjoint closed arcs $I_j \subset \mathbf{RP}^1$ such that $A_j \subset I_j$ for $1 \le j \le q$. Our main result is

Theorem 1 *Let $\{A_j\}_{j=1}^q$ be a separated collection of finite sets in $\mathbf{RP}^1$, each set A_j containing $a_j + 1$ points, such that (1) holds, and let W_j be subspaces in $\mathbf{P}^d$ spanned by the sets $E(A_j)$. Then all subspaces X of codimension 2 which satisfy (2) are real. For a generic configuration of A_j there are exactly $K_{\boldsymbol{a}}$ such subspaces X.*

In general, the condition that A_j are separated cannot be removed. Let us consider several special cases.

1. Let $q = 2$, $a_1 = a_2 = d - 1$. Then the problem always has one solution,

this solution is real, and the condition that A_j are separated is redundant.

2. Let us consider the limiting situation when all points in each A_j collide. Here $a_j = \operatorname{card} A_j - 1$ are arbitrary integers satisfying (1). Then W_j are subspaces osculating E at some real points; this case was considered in [1, 2]. We will show that this situation is generic enough and the number of solutions is exactly $K_{\boldsymbol{a}}$, for *any* choice of the real points (Theorem 3 below). If $\boldsymbol{a} = (1, \ldots, 1)$ then $K_{\boldsymbol{a}}$ is the Catalan number

$$K_{(1,\ldots,1)} = \frac{1}{d} \begin{pmatrix} 2d - 2 \\ d - 1 \end{pmatrix}. \tag{3}$$

3. Now we consider the case that $a_1 \in [1, d-1]$ is arbitrary and all the rest of a_j are equal to 1. The Kostka number in this case is

$$K_{(a_1,1,\ldots,1)} = \frac{a_1 + 1}{d} \begin{pmatrix} 2d - 2 - a_1 \\ d - 1 \end{pmatrix}. \tag{4}$$

In the case that the two points of each A_j, $j \geq 2$ collide, Problem 1 is equivalent to the following problem of enumeration of flags.

Consider flags $F = (F_1, F_2)$ in $\mathbf{P}^d$, where

$$F_2 \subset F_1 \quad \text{and} \quad \operatorname{codim} F_i = i, \quad i = 1, 2.$$

Suppose that $A_1 = \{x_0, \ldots, x_{a_1}\} \subset \mathbf{RP}^1$ and $A_j = \{y_j^{(2)}\}$, where the superscript 2 indicates that each point y_j is taken with multiplicity 2, $j = 2, \ldots, 2d - 2 - a_1$. Our condition on the flags F is that F_1 contains the points $E(x_k)$, $0 \leq k \leq a_1$, and F_2 intersects the tangent lines W_j to E at y_j, $j = 2, \ldots, 2d - 2 - a_1$. We want to enumerate such flags F.

The condition on F_1 implies that F_1 contains W_1, the subspace spanned by $E(x_k)$, $0 \leq k \leq a_1$. If x_k are in general position then $\dim W_1 = a_1$. Now $W_1 \subset F_1$ and $F_2 \subset F_1$ imply that

$$\dim W_1 \cap F_2 \geq a_1 - 1, \tag{5}$$

and the remaining conditions on F_2 are

$$\dim F_2 \cap W_j \geq a_j - 1 = 0, \quad j = 2, \ldots, 2d - 2 - a_1. \tag{6}$$

In the opposite direction, suppose that F_2 satisfies (5) and (6), and define F_1 as the span of F_2 and W_1. Then $\operatorname{codim} F_1 = 1$ and $F_2 \subset F_1$.

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Thus our problem of flag enumeration is equivalent to the problem of enumeration of subspaces F_2 of codimension 2 satisfying (5) and (6), which is a special limiting case of Problem 1. Assuming that the sets A_j are separated, Theorem 1 implies that all solutions are real, and for generic choice of points x_i and y_j the number of these solutions is given by (4). This problem of enumeration of flags was subject to extensive computer experiments [6].

In these experiments the role of the separation condition was discovered: when it holds all solutions are real, while when it does not hold, many configurations of points A_j give fewer real solutions than the upper estimate from Schubert calculus.

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2. Rational functions

The method of this paper is based on the relation between subspaces of codimension 2 and rational functions of one variable.

Let $G = G(d-1, d+1)$ be the Grassmannian of projective subspaces of dimension $d-2$ in $\mathbf{P}^d$. Every such subspace X can be defined by two equations in homogeneous coordinates

$$
\begin{aligned}
b_{0,0}z_0 + \ldots + b_{0,d}z_d &= 0, \\
b_{1,0}z_0 + \ldots + b_{1,d}z_d &= 0.
\end{aligned}
\tag{7}
$$

We put into correspondence to this system a rational function

$$
f(z) = \frac{b_{0,0} + b_{0,1}z + \ldots + b_{0,d}z^d}{b_{1,0} + b_{1,1}z + \ldots + b_{1,d}z^d}.
\tag{8}
$$

Choosing another system of equations of the form (7) that defines the same X results in replacing f by $\phi \circ f$, where ϕ is a rational function of degree 1. We will call rational functions f_1 and f_2 *equivalent* if $f_1 = \phi \circ f_2$, $\deg \phi = 1$. Let $D \subset G$ be the subset that corresponds to reducible rational functions (those whose numerator and denominator have a non-constant common factor). Then $G \backslash D$ is in bijective correspondence with equivalence classes of non-constant rational functions of degree at most d.

A rational function is called *real* if it maps $\mathbf{RP}^1$ into itself. An equivalence class is called real if it contains a real rational function. Real classes correspond to the real elements of G.

Now we translate conditions (2) to the language of rational functions. Let W be the subspace of dimension a spanned by the points $E(z_k)$, $0 \leq k \leq a$ on the rational normal curve. The condition that X given by (7) intersects W non-transversally means that

$$\operatorname{rank} BC = 1, \tag{9}$$

where

$$B = \begin{pmatrix} b_{0,0} & \cdots & b_{0,d} \\ b_{1,0} & \cdots & b_{1,d} \end{pmatrix}, \tag{10}$$

and C is the matrix whose columns are $(1, z_k, \ldots, z_k^d)$, $0 \leq k \leq a$. Equation (9) says that the two rows of BC are proportional, which is equivalent to

$$f(z_0) = f(z_1) = \ldots = f(z_a). \tag{11}$$

Thus in the case that each W_j is the span of $a_j + 1$ points on the rational normal curve, Problem 1 is equivalent to the following interpolation-type problem for rational functions:

Problem 1′. *For given subsets $A_j \subset \mathbf{P}^1$, of cardinality $a_j + 1$, where a_j satisfy (1), enumerate classes of rational functions f of degree d with the property that for every j, f is constant on A_j.*

Thus our Theorem 1 is equivalent to

Theorem 1′ *If the sets $A_j \subset \mathbf{RP}^1$ are separated then all solutions of Problem 1′ are real. For generic subsets $A_j \subset \mathbf{RP}^1$ the number of classes of real rational functions solving Problem 1′ is $K_{\boldsymbol{a}}$.*

It is not known whether the genericity condition in the second statement of Theorem 1′ can be removed. It can be removed in the following special case.

Theorem 2 *Let $A_1, \ldots, A_q$ be separated sets on the real line, $\operatorname{card} A_j = a_j + 1 \in [2, d]$, and*

$$\sum_{j=1}^{q} a_j = d - 1.$$

Then there exists a unique class of polynomials p of degree d, satisfying the conditions that p is constant on A_j for $1 \leq j \leq q$.

Proof. We restrict ourselves for simplicity to the case that $q = d - 1$ and $a_j = 1$ for $1 \leq j \leq q$. The general case is similar. Let $A_j = \{x_j, y_j\}$, where

$$x_1 < y_1 < x_2 < y_2 < \ldots < x_{d-1} < y_{d-1}. \tag{12}$$

Every polynomial of degree at most d is equivalent to a unique polynomial of the form

$$p(z) = z^n + \ldots + b_1 z, \quad \text{where} \quad n \leq d. \tag{13}$$

Conditions of the theorem mean that

$$p(x_j) = p(y_j), \tag{14}$$

for $1 \leq j \leq d - 1$. This is a system of linear equations with respect to coefficients of p, and it is easy to see that the coefficients of this system are real. Thus the system (14) has unique solution of the form (13) if and only if it has a unique real solution of this form.

Now suppose that our system (14) has two solutions of the form (13). Subtracting these solutions we obtain another solution which is a real polynomial p_0 of degree strictly less than d, and

$$p_0(0) = 0. \tag{15}$$

On the other hand, Rolle's theorem and (14) implies that the derivative p_0' has at least $d - 1$ zeros. So $p_0' = 0$ and we conclude from (15) that $p_0 = 0$. This proves the theorem. $\square$

In the limit, when in each group A_j all points collide to one point x_j we recover from Theorem $1'$ the main result of [1]: *all rational functions whose critical points are real are equivalent to real rational functions.*

In this degenerate case we have an important additional information:

Theorem 3 *The number of classes of rational functions of degree d with arbitrarily prescribed real critical points of given multiplicity is exactly $K_{\boldsymbol{a}}$, for any choice of the real critical points, where $\boldsymbol{a} = (a_1, \ldots, a_q)$ is the vector of multiplicities.*

Our proofs of Theorems $1'$ and 3 are based on the results in [1, 2] which we recall in the next section.

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3. Proof of Theorems 1′ and 3

It is sometimes convenient to replace $\mathbf{RP}^1$ by the unit circle $\mathbf{T}$ as in [1]. We always assume that $\mathbf{T}$ is equipped with the counter-clockwise orientation. We denote by $\mathbf{s} : \mathbf{P}^1 \to \mathbf{P}^1$ the reflection with respect to $\mathbf{T}$, that is $\mathbf{s}(z) = 1/\bar{z}$.

We fix $d \geq 2$. Let R be the class of all rational functions f such that $f(\mathbf{T}) \subset \mathbf{T}$, all critical points of f are simple and belong to $\mathbf{T}$. We equip R with the following topology (see [2]): a sequence (f_k) converges to f if there exists a finite set $S \subset \mathbf{P}^1$ such that $f_k \to f$ uniformly on compact subsets of $\mathbf{P}^1 \backslash S$.

First we recall a parametrization of the class R. Let f be a function of the class R, and v_0 a critical point of f. The full preimage $\Gamma = f^{-1}(\mathbf{T})$ defines a cell decomposition of $\mathbf{P}^1$. We call the pair $\gamma = (\Gamma, v_0)$ the *net of f with respect to v_0*.

One can describe all objects which can occur in this way. Let Γ be a cell decomposition of $\mathbf{P}^1$ with the following properties:

(i) closure of each cell is homeomorphic to a closed ball of corresponding dimension,

(ii) Γ is symmetric with respect to $\mathbf{T}$, that is the reflection $\mathbf{s}$ maps each cell of Γ onto a cell of Γ,

(iii) all 0-cells belong to the unit circle $\mathbf{T}$ and there are $2d - 2$ of them,

(iv) 1-skeleton of Γ contains T.

As usual we call 0-, 1- and 2 cells vertices, edges and faces. The edges that do not belong to $\mathbf{T}$ will be called *interior edges*. A *net* is a pair $\gamma = (\Gamma, v_0)$ where Γ is a cell decomposition of $\mathbf{P}^1$ with the properties (i)-(iv) and v_0 is a vertex of Γ.

Two nets (Γ, v_0) and (γ', v_0') are called *equivalent* if there exists a homeomorphism ϕ of $\mathbf{P}^1$ preserving the orientations of both $\mathbf{P}^1$ and $\mathbf{T}$, commuting with the reflection $\mathbf{s}$ and having the properties $\phi(\Gamma) = \Gamma'$ and $\phi(v_0) = v_0'$.

One of the results of [1] is the following: *Let f_1 and f_2 be two rational functions in R with the same critical points, and v_0 one of these common critical points. If the nets of f_1 and f_2 with respect to v_0 are equivalent then f_1 and f_2 are equivalent.*

The main technical result of [1] can be stated as follows: *For every net γ whose vertex set is V, and for every injective map $c : V \to \mathbf{T}$ preserving*

the cyclic order induced by the orientation, there exists a unique class C of rational functions in R whose critical points are $c(V)$ and whose net with respect to $c(v_0)$ is equivalent to γ.

We will say somewhat informally that functions of this class C have prescribed net and prescribed critical points.

Functions $f \in R$ with a prescribed net and prescribed critical points can be normalized in the following way. Choose two additional vertices v_1 and v_{-1} and three distinct points w_{-1}, w_0, w_1 on $\mathbf{T}$. Then the normalization condition is

$$f(c(v_i)) = w_i, \quad i = -1, 0, 1. \tag{16}$$

There exists a unique normalized function $f \in R$ with a prescribed net and prescribed critical points. Another result from [1] says that for each fixed net, this normalized function depends continuously on the critical points, or more precisely, on the map c above.

Now we can prove Theorem 1'. Suppose that the separated sets $A_j \subset \mathbf{T}$ satisfying (1) are given. Let $I_j \subset \mathbf{T}$ be disjoint arcs containing A_j. We list each A_j as $(z_{j,0}, z_{j,1}, \ldots, z_{j,a_j})$ where the order within A_j is consistent with the counter-clockwise orientation of $\mathbf{T}$. For any two distinct points r and s in $\mathbf{T}$, we denote by $[r, s]$ the closed arc of $\mathbf{T}$ which is traced from r to s counter-clockwise.

Choose additional points

$$x_{j,k} \in [z_{j,k}, z_{j,k+1}], \quad 0 \le k \le a_j - 1, \quad 1 \le j \le q. \tag{17}$$

According to (1), the total number of these points $x_{j,k}$ is $2d - 2$. So we can use these points as vertices of a net. We consider the nets satisfying the following additional condition:

there are no interior edges connecting two vertices on the same I_j. (18)

Lemma 1 *Suppose that a net γ satisfies (18). Let all $x_{j,k}$, except one, $x = x_{j_0,k_0}$ be fixed arbitrarily, so that conditions (17) are satisfied. For fixed $w_i \in \mathbf{T}$, $j = 0, \pm 1$, let f_x be the function of the class R with the net γ and critical points $x_{j,k}$, normalized as in (16). Denote $I = [z_{j_0,k_0}, z_{j_0,k_0+1}]$. Then there exist a continuous branch of $\arg f_x(z)$, for $(x, z) \in I \times I$, such that the function*

$$\psi(x) = \arg f_x(z_{j_0,k_0}) - \arg f_x(z_{j_0,k_0+1})$$

changes sign as x runs over I.

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Proof. To simplify our notation we put $r = z_{j_0,k_0}, s = z_{j_0,k_0+1}$, so that $I = [r, s]$. Let ℓ be the interior edge of γ that lies in the unit disc and has x as an endpoint. Let v be another endpoint of this edge. We note that neither the assumptions nor the conclusions of the lemma depend on the choice of the normalization. So we may assume that v is one of the distinguished vertices, say $v = v_1$ in (16) and choose $w_1 = f_x(v) = -1$. Let D_1 and D_2 be two faces of the net γ in the unit disc which have ℓ as the common part of their boundaries. As the boundary of every face is mapped by f_x bijectively onto $\mathbf{T}$, and the vertex v is disjoint from I because of (18) we conclude that $f_x(z) \neq -1$ for $(x, z) \in I \times I$. Thus $\arg f_x(z)$ can be defined as a continuous function for $(x, z) \in I \times I$, with values in $(-\pi, \pi)$. Now, $z \mapsto \arg f_x(z)$ is strictly monotone in the opposite directions on the arcs $[r, x]$ and $[x, s]$: it is strictly increasing on one of these arcs and strictly decreasing on another one. This follows from the fact that one of the closed faces D_1 or D_2 is mapped homeomorphically onto the closed unit disc and another onto the closed exterior of the unit disc, while the arcs $[r, x]$ and $[x, s]$ are parts of the boundaries of D_1 and D_2.

We conclude that $\arg f_x(r) - \arg f_x(s)$ changes sign as x runs from r to s, which proves the lemma. $\qquad\square$

Lemma 2 *If γ is a net satisfying (18) then there exists a choice of critical points $x_{j,k}$ such that every function $f \in R$ with this net and with these critical points satisfies*

$$f(z_{j,0}) = \ldots = f(z_{j,a_j}) \quad \text{for every} \quad j \in [1, q]. \qquad (19)$$

Proof. Let $\mathbf{x}$ be the vector with coordinates $(x_{j,k})$. The set of $\mathbf{x}$ satisfying (17) is a closed cube Q. According to Lemma 1, the continuous functions $\psi_{j,k}(\mathbf{x}) = \arg f_{\mathbf{x}}(z_{j,k}) - \arg f_{\mathbf{x}}(z_{j,k+1})$ with appropriate choices of branches of arguments take values of opposite signs on the opposite facets of Q. It follows from a Corollary of Brouwer's Theorem [4, Ch. IV, 1D] that all these functions $\psi_{j,k}$ have a common zero in the interior of Q. $\qquad\square$

So, for each net γ satisfying (18), Lemma 2 gives at least one function $f \in R$ that satisfies (19). The functions with different nets cannot be equivalent. The crucial fact is

Lemma 3 *For each choice of the separated sets A_j with $\operatorname{card} A_j = a_j + 1$ there are exactly $K_{\mathbf{a}}$ nets satisfying (18).*

Proof. We describe two injective maps: one from the set of nets to the

set of SSYT (part A below) and another one in the opposite direction (part B).

First of all we observe that if two nets (Γ, v_0) and (γ', v_0') satisfying (18) are equivalent then v_0 and v_0' cannot belong to the same interval I_j. Therefore, instead of distinguishing a vertex v_0, it is sufficient to distinguish one of the intervals I_j. This is achieved by considering the real line instead of the unit circle $\mathbf{T}$, which distinguishes the leftmost interval.

We assume for convenience that the intervals are enumerated in such a way that $i < j$ implies $x < y$ for all $x \in I_i$ and $y \in I_j$, besides the vertices in each interval are enumerated from left to right.

Part A. We construct an SSYT corresponding to a given net. To each vertex of the net we assign one entry in SSYT. These entries are defined inductively: for $n \geq 1$, suppose that the entries corresponding to x_k for $k \leq n - 1$ are already placed in the tableau. The entry corresponding to x_n is the number j such that $x_n \in I_j$. This number j is placed in the first row if the edge from x_n goes forward (to some $x_m > x_n$) and in the second row if the edge from x_n goes backward. The new entry j is placed to the leftmost place in its row. It is easy to see that in this way we obtain an SSYT for each net satisfying (18).

Part B. In the opposite direction, we construct inductively the net corresponding to a given SSYT of shape $2 \times (d - 1)$. On the n-th step of the construction we deal with a tableau of shape $2 \times (d - n)$ filled by positive integers such that the rows are non-decreasing and columns are strictly increasing. The only difference from SSYT is that the integers in the tableau are not necessarily consequtive; slightly abusing terminology we will call these tableaux SSYT as well. Assuming that such an SSYT is given, let k be the first (leftmost) entry in its second row. We find the rightmost entry m in the first row that is strictly less than k and draw an edge between the leftmost free vertex in I_k and the rightmost free vertex of I_m. Then we delete the first entry from the second row and the above defined entry from the first row, and shift the left part of the first row to the right to form a tableau of shape $2 \times (d - n - 1)$. Evidently, the obtained tableau is an SSYT, so we can proceed until it becomes empty.

To complete Part B, it remains to prove that the edge added on the n-th step does not intersect the edges added on the previous steps.

Let us show first that on each step all the vertices lying between the endpoints of the added edge are not free. Indeed, suppose that there is a free

vertex v between the endpoints. We use the notation from the description of the n-th step above. By assumption, the vertex v belongs to some I_p with $m < p < k$, since the new edge joins the leftmost free vertex in I_k to the rightmost free vertex in I_m. Since the vertex v in I_p is free there should be an entry p somewhere in the current SSYT. It cannot be in the second row since $k > p$ is the leftmost entry of the second row. It cannot be in the first row since $m < p$ is the rightmost entry of those which are strictly less than k.

Now we can complete the proof. If a new edge intersects am old one, then one of the vertices of the new edge was a free vertex between the endpoints of the old edge on the step when this old edge was added. This completes the proof of the part B. $\square$

Now we can complete the proof of Theorem 1′. For given sets A_j in general position, Schubert calculus gives at most $K_{\boldsymbol{a}}$ classes of rational functions solving Problem 1′. On the other hand, Lemmas 2 and 3 give at least $K_{\boldsymbol{a}}$ real classes of such functions. Thus all classes are real. Passing to the limit, we obtain that for all separated sets A_j, all solutions of Problem 1′ are real. $\square$

Proof of Theorem 3. Let R' be the subset of the closure of R which consists of functions of degree exactly d. The nets of functions $f \in R'$ have similar properties to those of $f \in R$, except that the vertex degrees can be arbitrary numbers between 4 and $2d$. Such nets were called degenerate in [2]. In that paper, the following result was proved: *Let $x_1^{a_1}, \ldots, x_q^{a_q}$ be a divisor on the real line, where $\boldsymbol{a} = (a_1, \ldots, a_q)$ satisfies (1) and γ be a net whose vertices enumerated in the increasing order have degrees $2a_j + 2$. Then there exists a real rational function of degree d with critical points of multiplicity a_k at the points x_k.*

It is easy to see that the number of these degenerate nets γ is the same as the number of non-degenerate nets satisfying (18), so according to Lemma 3, this number is $K_{\boldsymbol{a}}$. Hence the number of classes of real functions satisfying the conditions of Theorem 3 is at least $K_{\boldsymbol{a}}$ and this proves the theorem. $\square$

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